Graph Representations for Bunched Implications

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1 Introduction

Cographs were discovered independently by several authors and classified by their P_4-freeness [5]. Built from operations mimicking operations from Boolean algebra, they provide a faithful graph-theoretic representation of classical propositional formulas. In [3], Dominic Hughes initiated the study of combinatorial proofs by leveraging this graph-theoretic representation to fully account for classical propositional proofs. There are now quite a few papers in the area of combinatorial proofs extending Hughes' system to first-order classical logic [4], various modal logics [1], and the multiplicative fragment of intuitionistic logic [2].

While combinatorial proof theory is interesting for its own sake, it is also philosophically motivated. With every combinatorial proof comes the range of sequent calculus proofs that can be used to construct it, and in this way one can see combinatorial proofs structures as equivalence classes of proofs. By way of equating proofs of similar structural complexity, combinatorial proof theory constitutes a meaningful contribution to Hilbert's 24th problem [9].

A family of logics for which a quantitative approach to proof simplicity would be especially impactful is that of the bunched implication logics of Peter O'Hearn and David Pym [6]. These logics are widely used to reason about resource-sensitive software [8], and automating proof simplification in this family could have applications to software verification [7]. To the author's knowledge, no combinatorial proof system has been devised for even the simplest logic in this family, propositional bunched implication logic pBI.

In what follows, a first step towards a combinatorial proof system for pBI is taken. A graph-theoretic representation of formulas in MMBI, the so-called magic multiplicative fragment of pBI, is given in section 2.

The representation given is heavily inspired by those found in the four papers mentioned in the first paragraph [3], [2], [4], but especially [2]. After the representation is defined, a list of basic properties enjoyed by MMBI-constructed graphs is recorded in section 3. A structural characterization of MMBI-constructed formulas appears in section 4. A brief account of how to extend the representation appears in section 5, with further reconstruction theorems found in **??**.

2 Basic Definitions and Notation

The *magic multiplicative* fragment MMBIof pBI is composed of the expressions generated recursivley by

$$\mathcal{F} ::= p \in \mathbf{P} \mid \mathcal{F} \wedge \mathcal{F} \mid \mathcal{F} \Rightarrow \mathcal{F} \mid \mathcal{F} * \mathcal{F} \mid \mathcal{F} \mathrel{-\!\!*} \mathcal{F},$$

where $\mathbf{P}$ is some fixed set of propositional variables. The goal of this document is to represent MMBI-formulas with so-called $(1,2)$-*mixed directed acyclic graphs*, and to characterise those graphs that represent MMBI-formulas in structural terms.

Definition 2.1. A *mixed dag* is a triple $(G, \frown, \rightharpoondown)$, where $\frown$ is a symmetric irreflexive relation on G whose pairs are called *undirected edges*, and the *underlying dag* $(G, \rightharpoondown)$ is a directed acyclic graph. Fix a pair of families of colours I_u, I_d with $|I_u| = n$ and $|I_d| = m$. An (n,m)-*mixed dag* is a mixed dag whose undirected edges have been assigned colours in I_u and directed edges have been assigned colours in I_d.

The typical abuse of notation applies: If the (n,m)-mixed structure of a mixed graph $(G, \frown, \rightharpoondown)$ is clear, reference may only be made to G. Given a set of propositional variables $\mathbf{P}$, an $\mathbf{P}$-*labelling* of G is simply an assignment $V(G) \to \mathbf{P}$.

Remark. Only the case $n = 1$, $m = 2$ is necessary for representing MMBI-formulas. However, I conjecture that a similar reconstruction theorem to that of section 4 is likely within reach for full pBI in the setting of $(2,2)$-mixed graphs, as well as for l-dimensional versions of MMBIand pBI in the setting of (l,l)-mixed graphs. See section 5 for more details.

Let G be an arbitrary (n,m)-mixed dag, and let $v \in G$. The following are some basic terminological and notational conventions adopted from here on out.

1. The set of vertices accessible to v is denoted $(v\rightharpoondown) = \{w \mid v \rightharpoondown w\}$.

2. The set of vertices n-accessible to v is defined recursively by $(v\rightharpoondown^0) = \{v\}$ and $(v\rightharpoondown^{n+1}) = \{w \mid \exists x \in (v\rightharpoondown^n). x \rightharpoondown w\}$ and
$$(v\rightharpoondown^*) = \bigcup_{n=1}^{\infty} (v\rightharpoondown^n).$$
It is also convenient to define $(v\rightharpoondown^{-1}) = \varnothing$. Note also that, due to acyclicity, $v \notin (v\rightharpoondown^*)$.

3. A modest addition to the previous definition gives the *cone* of v, defined $\mathrm{Cone}(v) = \{v\} \cup (v\rightharpoondown^*)$. The cones of a given mixed graph often provide an intuitive geometrical representation of its structure.

4. For an induced subgraph Y of G, define $(Y\rightharpoondown) = \bigcup\{(y\rightharpoondown) \mid y \in Y\}$, and similarly $\mathrm{Cone}(Y) = \bigcup\{\mathrm{Cone}(y) \mid y \in Y\}$.

5. If $(v\rightharpoondown) = \varnothing$, then v is called a *root*.

6. If $w \rightharpoondown v$ for no $w \in G$, then v is called a *leaf*.

7. The set of roots of G is denoted $\sqrt{G}$.

8. The set of nth roots of G is defined inductively by

$$\sqrt[n]{G} = \sqrt{G - \bigcup_{0<k<n} \sqrt[k]{G}}.$$

These provide a somewhat generative picture for G that will become useful in the characterization process in section 3.

9. The graph G is *root connected* if for any two $v, w \in \sqrt{G}$ there is a cone $\mathrm{Cone}(a)$ containing both v and w.

10. The *depth* $\mathrm{depth}(v)$, when it is well-defined, denotes the largest n for which there is a path $v \curvearrowright x_1 \curvearrowright \cdots \curvearrowright x_n$ such that x_n is a root.

For $(1,2)$-mixed dags, only the colour blue is used for undirected edges, and black and pink are used for directed edges. The following notation is used.

$$\begin{aligned}
V(G) &= G \\
{\color{blue}\frown}(G) &= \frown(G) \\
\curvearrowright(G) &= (\text{directed edges of } G \text{ coloured black}) \\
{\color{magenta}\curvearrowright}(G) &= (\text{directed edges of } G \text{ coloured } {\color{magenta}\textit{pink}})
\end{aligned}$$

This also gives ${\color{blue}\frown}$ and ${\color{magenta}\curvearrowright}$ interpretations as binary relations.

The graph operations relevant to $\mathsf{MMBI}$are defined as follows. Let H and K be arbitrary $(1,2)$-mixed dags.

- The *disjoint union*, or *sum*, $G = H \sqcup K$ of H and K is defined by

$$\begin{aligned}
V(G) &= V(H) \sqcup V(K) \\
\curvearrowright(G) &= \curvearrowright(H) \sqcup \curvearrowright(K) \\
{\color{magenta}\curvearrowright}(G) &= {\color{magenta}\curvearrowright}(H) \sqcup {\color{magenta}\curvearrowright}(K) \\
{\color{blue}\frown}(G) &= {\color{blue}\frown}(H) \sqcup {\color{blue}\frown}(K).
\end{aligned}$$

- The *connect* $G = H \mathbin{\color{blue}||} K$ of H and K is defined by

$$\begin{aligned}
V(G) &= V(H) \sqcup V(K) \\
\curvearrowright(G) &= \curvearrowright(H) \sqcup \curvearrowright(K) \\
{\color{magenta}\curvearrowright}(G) &= {\color{magenta}\curvearrowright}(H) \sqcup {\color{magenta}\curvearrowright}(K) \\
{\color{blue}\frown}(G) &= {\color{blue}\frown}(H) \sqcup {\color{blue}\frown}(K) \sqcup \{v \mathbin{\color{blue}\frown} w \mid v \in H \text{ and } w \in K\}.
\end{aligned}$$

- The *subjunction* $G = H \blacktriangleright K$ of H and K is defined by

$$\begin{aligned}
V(G) &= V(H) \sqcup V(K) \\
\curvearrowright(G) &= \curvearrowright(H) \sqcup \curvearrowright(K) \sqcup \{v \curvearrowright w \mid v \in \sqrt{H} \text{ and } w \in \sqrt{K}\} \\
{\color{magenta}\curvearrowright}(G) &= {\color{magenta}\curvearrowright}(H) \sqcup {\color{magenta}\curvearrowright}(K) \\
{\color{blue}\frown}(G) &= {\color{blue}\frown}(H) \sqcup {\color{blue}\frown}(K) \sqcup \{v \mathbin{\color{blue}\frown} w \mid v \in H \text{ and } w \in K\}.
\end{aligned}$$

- The *(magic-)connected subjunction* $G = H \mathbin{{\color{blue}|}{\color{magenta}\blacktriangleright}} K$ of H and K is defined by

$$\begin{aligned}
V(G) &= V(H) \sqcup V(K) \\
\curvearrowright(G) &= \curvearrowright(H) \sqcup \curvearrowright(K) \\
{\color{magenta}\curvearrowright}(G) &= {\color{magenta}\curvearrowright}(H) \sqcup {\color{magenta}\curvearrowright}(K) \sqcup \{v \mathbin{\color{magenta}\curvearrowright} w \mid v \in \sqrt{H} \text{ and } w \in \sqrt{K}\} \\
{\color{blue}\frown}(G) &= {\color{blue}\frown}(H) \sqcup {\color{blue}\frown}(K) \sqcup \{v \mathbin{\color{blue}\frown} w \mid v \in H \text{ and } w \in K - \sqrt{K}\}.
\end{aligned}$$

In each case, if H and K are $\mathbf{P}$-labelled, then the $\mathbf{P}$-labelling of $H \square K$ is the sum of the two $\mathbf{P}$-labellings, where $\square$ is any of the operations listed above.

Given a propositional variable p, define

$$\mathscr{G}(p) = (\{v^{(p)}\}, \varnothing, \varnothing, \varnothing),$$

where the parenthetical superscript denotes the $\mathcal{A}$-labelling. Then, given two MMBI-formulas φ and ψ,

$$\begin{aligned}
\mathscr{G}(\varphi \wedge \psi) &= \mathscr{G}(\varphi) \sqcup \mathscr{G}(\psi) \\
\mathscr{G}(\varphi \Rightarrow \psi) &= \mathscr{G}(\varphi) \blacktriangleright \mathscr{G}(\psi) \\
\mathscr{G}(\varphi * \psi) &= \mathscr{G}(\varphi) \,||\, \mathscr{G}(\psi) \\
\mathscr{G}(\varphi \mathrel{-\!\!*} \psi) &= \mathscr{G}(\varphi) \,|\!\blacktriangleright \mathscr{G}(\psi).
\end{aligned}$$

A $\mathbf{P}$-labelled $(1,2)$-mixed dag G is called *MMBI-constructed* if there is an MMBIformula φ for which $\mathscr{G}(\varphi) = G$. An unlabelled $(1,2)$-mixed dag is called *MMBI-constructed* if it admits a $\mathbf{P}$-labelling for which it is MMBI-constructed.

In MMBI, there are some canonical equivalences between formulas that should be reflected in the construction of our graphs. In particular, the $*$ and $\wedge$ operators of bunched implication logic are interpeted category-theoretically as parallel symmetric monoidal structures. With this in mind, define the following congruence relation $\sim$ generated by the pairs

$$\begin{aligned}
\varphi \wedge \psi &\sim \psi \wedge \varphi, \\
\varphi * \psi &\sim \psi * \varphi, \\
(\varphi \wedge \psi) \wedge \chi &\sim \varphi \wedge (\psi \wedge \chi), \\
(\varphi * \psi) * \chi &\sim \varphi * (\psi * \chi), \\
(\varphi \wedge \psi) \Rightarrow \chi &\sim \varphi \Rightarrow (\psi \Rightarrow \chi), \\
(\varphi * \psi) \mathrel{-\!\!*} \chi &\sim \varphi \mathrel{-\!\!*} (\psi \mathrel{-\!\!*} \chi),
\end{aligned}$$

so that $\sim$ is an equivalence relation satisfying

$$\begin{aligned}
\varphi \wedge \psi &\sim \chi \wedge \eta \\
\varphi * \psi &\sim \chi * \eta \\
\varphi \Rightarrow \psi &\sim \chi \Rightarrow \eta \\
\varphi \mathrel{-\!\!*} \psi &\sim \chi \mathrel{-\!\!*} \eta
\end{aligned}$$

when $\varphi \sim \chi$ and $\psi \sim \eta$. Let $\sim_{\mathscr{G}}$ be the equivalence relation defined so that $\varphi \sim_{\mathscr{G}} \psi$ iff $\mathscr{G}(\varphi) = \mathscr{G}(\psi)$. If $\mathscr{G}$ is to respect the category-theoretic interpretation of $*$ and $\wedge$, the identity $\sim_{\mathscr{G}} = \sim$ should hold. Left-to-right inclusion is shown in the next section, but right-to-left inclusion won't appear until section 4.

The reader might wonder whether $\mathscr{G}$ was designed specifically to satisfy the equality $\sim = \sim_{\mathscr{G}}$, and indeed that is correct. The function $\mathscr{G}$ is a marrying of the graph-constructors used in [2] and [3], the first of was designed to respect the closed interpretation of $\wedge$ in multiplicative linear logic, and the other to respect the cartesian structure of $\wedge$ in classical logic.

3 Basic Properties of MMBI- Generated Graphs

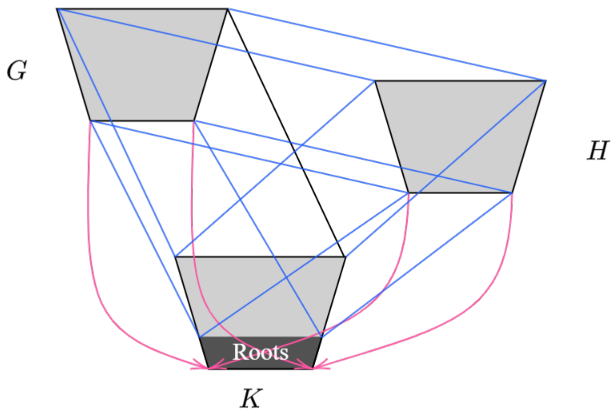
What follows is a string of lemmas that will be useful in the characterization of MMBI-constructed $(1,2)$-mixed dags. The explicit $\mathbf{P}$-labelling of a given graph will often elude explicit exposition in what follows, mostly because it will not play a significant role.

Lemma 3.1. $\sim \subseteq \sim_{\mathscr{G}}$

Proof. It suffices to check that $\sqcup$, $\parallel$, $\blacktriangleright$, $|\blacktriangleright$ satisfy the equations generating $\sim$. It is easily checked that $\sqcup$ and $\parallel$ are commutative and associative, as well as the identity

$$(H \sqcup K) \blacktriangleright L = H \blacktriangleright (K \blacktriangleright L).$$

To see that the analogous identity holds for $\parallel$ and $|\blacktriangleright$, it suffices to stare at the following picture for a moment or two.



$\square$

See corollary 4.16 for the converse.

For the rest of this section, fix a MMBI-constructed $(1, 2)$-mixed dag G.

Lemma 3.2. *The underlying dag of G is both L-**free** and Σ-**free**, meaning that none of its induced subgraphs are of either of the forms below.*

$$\begin{tikzcd} z \arrow[dd] \arrow[dr] & \\ & y \arrow[dl] \\ x & \end{tikzcd} \qquad \begin{tikzcd} & z \arrow[dl] \arrow[dr] & \\ w & & y \arrow[dl] \\ & x & \end{tikzcd} \qquad \begin{tikzcd} w \arrow[d] \arrow[dr] & & y \arrow[d] \arrow[dl] \\ u & z & x \end{tikzcd}$$

The first two graphs are L**-shaped** graphs, and the third graph is a Σ**-shaped** graph.

Proof. This is more a result about pure dags than about mixed dags, of course. See [2] for details. $\square$

Lemma 3.3. *Consider the underlying dag $(G, \rightharpoonup)$ of G, and define the simple graph G^* by setting*

$$\begin{aligned} V(G*) &= G, \\ \frown(G^*) &= \{v \frown w \mid (v \rightharpoonup w) \vee (w \rightharpoonup v)\}. \end{aligned}$$

Then G^* is connected if and only if G is root connected.

Proof. It is clear that if G is root connected, then G^* is connected. Going in the other direction, suppose G is not root connected and let $v, w \in \sqrt{G}$ such that $v, w \in \mathrm{Cone}(a)$ for no $a \in G$. I will aim to show that there is no path $v \frown x_1 \frown \cdots \frown x_n = w$. We proceed by induction on n. If $v \frown w$, then either $v \rightharpoonup w$ or $w \rightharpoonup v$, contradicting $v, w \in \sqrt{G}$. Now suppose there is no path of length n between v and w, but that

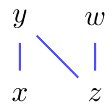
$$x_0 = v \frown x_1 \frown \cdots \frown x_{n+1} = w$$

for some $x_1, \ldots, x_n$. Let m be the least index at which $x_m \curvearrowright x_{m+1}$. Since v, w are roots, $0 < m \leqslant n$. By L-freeness, $x_{m+1} \curvearrowright x_{m-1}$, making

$$v \frown x_1 \frown \cdots \frown x_{m-1} \frown x_{m+1} \frown \cdots w$$

a path of length n. By the induction hypothesis, no such path can exist. Hence, no path exists between v and w, putting v and w in different connected components of G^*. $\square$

Lemma 3.4. *G is N-*free*, meaning that it is free of induced subgraphs of the form*



Proof. By induction on the construction of G. Let the N-shaped graph appear as a subgraph of $G = H \mathbin{||} K$, and assume H and K are N-free MMBI-constructed graphs. There are two possible ways x, y, w, z could be partitioned into H and K:

- In the first, one of the four vertices is in one of H or K, and the remaining three are in the other. Given this partition, however, one of x, y, z, w enjoys full edge connections with the other three vertices.
- In the second possibility, two of the vertices are in one set, and the remaining two are in the other. Given this partition, two of the vertices enjoy full edge connections with the other two.

In either case, this N-shaped graph does not appear as an induced subgraph of G. $\square$

Noteworthy in the above lemma is the lack of directed edges in an N-shaped graph: This lemma does *not* ensure that $(G, \frown)$ is an N-free graph. What it does tell us is that the roots of a MMBI-constructed graph induce an N-free graph, and more generally that any subgraph without directed edges must be N-free as well. lemma 3.6 states precisely this observation below.

Lemma 3.5. *G has* full web connections, *meaning that if H and K are root connected components of $G - \sqrt[n]{G}$, either H and K share no edges in $(G, \frown)$ or they enjoy full edge connections in $(G, \frown)$.*

Proof. By induction on the construction of G. Clearly $\mathscr{G}(p)$ has full web connections. So, assume H and K are MMBI-constructed graphs with full web connections, and consider the following possibilities.

1. If $G = H \sqcup K$, then H and K share no $\frown$ edges at all. Since H and K have full web connections, G has full web connections.

2. If $G = H \mathbin{||} K$, then H and K enjoy full $\frown$ connections. This means that every root connected component of $H - \sqrt[n]{H}$ enjoys full $\frown$ connections with every root connected component of $K - \sqrt[n]{K}$. This gives G full web connections.

3. Suppose $G = H \blacktriangleright K$. We have
$$G - \sqrt{G} = H \sqcup (K - \sqrt{K}),$$
and H and $K - \sqrt{K}$ have full web connections. This implies $G - \sqrt{G}$ has full web connections. Since G is root connected, G has full web connections.

4. Suppose $G = H \mathbin{|\!\blacktriangleright} K$. We have
$$G - \sqrt{G} = H \mathbin{||} (K - \sqrt{K}),$$
and H and $K - \sqrt{K}$ have full web connections. This implies $G - \sqrt{G}$ has full web connections. Again, G is root connected, so G has full web connections.

$\square$

Lemma 3.6. *The set of weakly connected components of $G - \sqrt[n]{G}$ form an N-free simple graph,*

$$V(\mathcal{K}_n(G)) = \{H \mid H \textit{ is a weakly connected component of } G - \sqrt[n]{G}\}$$
$$\frown(\mathcal{K}_n(G)) = \{H \frown H' \mid (\exists v \in H)(\exists w \in H')v \frown w \textit{ in } G\},$$

for any n.

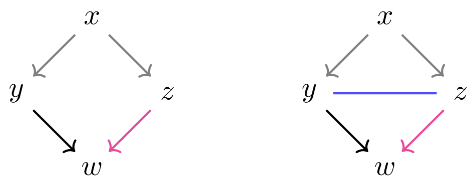
Proof. Suppose there are distinct root connected components H_i of $G - \sqrt[n]{G}$ such that $H_1 \frown H_2 \frown H_3 \frown H_4$ appears in $\mathcal{K}_n(G)$. By lemma 3.5, there are vertices $v_i \in H_i$ such that $v_1 \frown v_2 \frown v_3 \frown v_4$ in G. However, since the v_i are in distinct H_i, there are no directed edges of the form $v_i \rightharpoondown v_j$ in G. Since G is N-free, we must have $v_i \frown v_j$, and therefore $H_i \frown H_j$, for some $i, j = 1, \ldots, 4$. □

In particular, define $\mathcal{K}(G) = \mathcal{K}_0(G)$ to be the simple graph built from the weakly connected components of G.

Lemma 3.7. *G is* split-variation-free, *meaning that no vertex of G is both the source of a normal edge and a magic edge.*

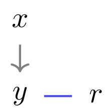
Proof. Again by induction on the construction of G. Let H and K be split-variation-free MMBI-constructed graphs and suppose x is a root of H. If $G = H \blacktriangleright K$, then $x \rightharpoondown y$ for any $y \in \sqrt{K}$, and there are no other directed edges with source x. Since H and K are split-variation-free, and every edge in G that is neither in H nor K is of this form, G is split-variation-free. The case for $G = H \mid\!\blacktriangleright K$ case is similar. □

Lemma 3.8. *G is* join-variation-free, *where a* join-variation *is an induced subgraph of one of the forms*



Proof. By induction on the complexity of G. Assume H and K are join-variation-free MMBI-constructed graphs, and suppose either $G = H \blacktriangleright K$ or $G = H \mid\!\blacktriangleright K$. Suppose further that the above join-variation appears as a subgraph of G, and assume without loss of generality that $x \in H$. If y and z are in K, they are by assumption non-roots, so y and z must be in H as well. Now, if $G = H \blacktriangleright K$ and $w \in K$, then $z \rightharpoondown w$, contradicting our assumption that $z \rightharpoondown w$ in G. This puts $w \in H$, which contradicts our assumption that H is join-variation-free. A similar argument applies to the case in which $G = H \mid\!\blacktriangleright K$. □

Lemma 3.9. *G is free of* dangling roots, *where a* dangling root *is an induced subgraph of G of the form*



where r is a root in G.

Proof. Again, by induction on the complexity of G. Assume H and K are MMBI-constructed graphs that are free of dangling roots. Suppose $G = H \parallel K$, and that the above appears as a subgraph of G. If either x or z appears in H or K, then they must both appear in H or K respectively. Thusly, the only interesting scenario to consider is the one in which $x, y \in H$ and $r \in K$. In such a case, $x \frown r$ and the above graph does not appear as an induced subgraph of G.

Next, suppose $G = H \blacktriangleright K$ or $G = H \mid\!\blacktriangleright K$. The only interesting scenario to consider, here, is the one in which $x \in H$ and $y, r \in K$. However, since r is a root, $x \rightharpoondown r$ in G. □

Lemma 3.10. *G is* ladder-free*, where a* ladder *is an edge $v \frown w$ such that $w \in \mathrm{Cone}(v)$.*

Proof. Let $v \rightharpoonup v_1 \rightharpoonup \cdots \rightharpoonup v_n \rightharpoonup w$ be a path in G. That there is no edge $v \frown w$ follows by induction on the construction of G. Suppose the lemma holds for MMBI-constructed graphs H and K, and that $G = H \blacktriangleright K$ or $G = H \mid\!\blacktriangleright K$. We can safely assume $w \in K$. Since there is at most one edge between any two vertices, $n > 0$. We can assume $v_n \in H$, for $v_i \in K$ implies $v_{i-1} \in K$ for all i since no v_i is a root of K. However, if w is a root of K and $w \frown v$ in G, then $v \in K$ as well. This implies that $v_i \in K$ for all i. $\square$

Lemma 3.11. *G is free of* trailing roots*, where a* trailing root *is an induced subgraph of the form*

$$\begin{tikzcd}
x \arrow[d] \arrow[dddr, dash] & \\
\vdots \arrow[d] & \\
r & s
\end{tikzcd}$$

with r and s roots of G.

Proof. By induction on the construction of G. Let H and K be MMBI-constructed graphs free of trailing roots, and assume that the above graph appears in G. If $G = H \parallel K$, the only interesting possibility to consider is that the path $x \rightharpoonup^* r$ is in one of H, K, and s is in the other. In this case, however, there is an edge $s \frown r$.

Next, suppose $G = H \blacktriangleright K$ or $G = H \mid\!\blacktriangleright K$. In either case, the interesting possibility to consider is the one in which an initial segment of the path $x \rightharpoonup \cdots \rightharpoonup r$ appears in H and $r \in K$. In this case, because s is a root, we must have $s \in H$ as well, but this would put $s \rightharpoonup r$ in G. $\square$

Lemma 3.12. *G is* box-free*, where a* box*-shaped graph is an induced subgraph of G of any of the forms*

$$\begin{array}{ccc}
\begin{array}{ccc} x & & z \\ \downarrow & & \downarrow \\ y & — & w \end{array}
&
\begin{array}{ccc} x & — & z \\ \downarrow & & \downarrow \\ y & & w \end{array}
&
\begin{array}{ccc} x & — & z \\ \downarrow & & \downarrow \\ y & — & w \end{array}
\\[1ex]
(i) & (ii) & (iii)
\end{array}$$

Proof. By induction on the complexity of G. Assume H and K are box-free, and that any of the above graphs appear as a subgraph of G. If $G = H \parallel K$, the only interesting case to consider is the one in which $x, y \in H$ and $z, w \in K$ (or vice versa). In this case, there is an edge $x \frown w$ in G.

Suppose $G = H \blacktriangleright K$ or $G = H \mid\!\blacktriangleright K$. If $x, z \in H$ and $y, w \in K$, then every vertex is a root and there is an edge $x \rightharpoonup w$ in G.

There is one interesting case remaining: $x \in H$ and $z, w, y \in K$. Note that here, y is a root and w is not. In (i) and (iii), if there are no additional edges, the subgraph of K induced by y, w, and z is a dangling root of G. Since G is MMBI-constructed, this contradicts the previous lemma. In (ii), since $x \frown z$, it must be the case that $G = H \mid\!\blacktriangleright K$. However, since w is not a root, there must also be an edge $x \frown w$. $\square$

Lemma 3.13. *G is $\boxslash$-free, where a $\boxslash$-shaped graph is a graph of the form*

$$\begin{tikzcd}
x \arrow[r, dash] \arrow[d] & y \arrow[d] \arrow[dl, dash] \\
z \arrow[r, dash] & w
\end{tikzcd}$$

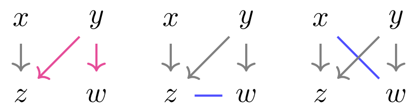
Proof. By induction on the complexity of G. Assume H and K are $\boxslash$-free, and that the above graph appears as a subgraph of G. If $G = H \parallel K$, the only interesting case to consider is the one in which $x, y \in H$ and $z, w \in K$ (or vice versa). In this case, there is an edge $x \frown w$ in G.

Now suppose that $G = H \blacktriangleright K$ or $G = H \mathrel{|\!\blacktriangleright} K$. There are three cases to consider.

1. If $y \in H$ and $x, z, w \in K$, then z is a root and y, z, w form a dangling root in K.
2. If $x, y \in H$ and $z, w \in K$, then $x \rightharpoonup w$.
3. Suppose $x \in H$ and $y, z, w \in K$. If $G = H \blacktriangleright K$, then there is no edge $x \frown y$. If $G = H \mathrel{|\!\blacktriangleright} K$, then since w is not a root, $x \frown w$.

In any case, the above graph does not appear as an induced subgraph of G. $\square$

Lemma 3.14. *G is free of* dangling sources*, where a vertex x is a* dangling source *if it appears as an induced subgraph of one of the forms*

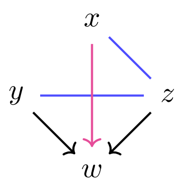


(i) (ii) (iii)

Proof. By induction on the complexity of G. Let H and K be MMBI-constructed graphs free of dangling sources, and suppose G is either $H \blacktriangleright K$ or $H \mathrel{|\!\blacktriangleright} K$ and that (i),(ii), or (iii) appears in G. If $x \in H$ and $y, z, w \in K$, then since z is a root and w is not, y, z, w, and any $v \in (w \rightharpoonup)$ induce an L-shaped graph in K. Now suppose $x, y \in H$ and $z, w \in K$. This makes both z and w roots of K, putting $x \rightharpoonup w$ in G.

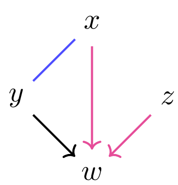
Finally, assume $y \in H$ and the rest of the vertices are in K. In this case, z, w are roots. For the graph (i), the interesting case is $G = H \mathrel{|\!\blacktriangleright} K$. In this scenario, $y \frown x$ in G. In the graph (ii), if $G = H \blacktriangleright K$, then a dangling root is induced by x, z, w in K. If $G = H \mathrel{|\!\blacktriangleright} K$ instead, then $y \frown x$ in G. Finally, consider the graph (iii). If $G = H \mathrel{|\!\blacktriangleright} K$ or $G = H \blacktriangleright K$, then x, z, w form a trailing root in K. $\square$

Lemma 3.15. *G is* slice-variation-free*, where a* slice-variation *is an induced subgraph of the form*



Proof. Again, by induction on the complexity of G. Suppose H and K are MMBI-constructed slice-variation-free graphs, and suppose $G = H \blacktriangleright K$ or $G = H \mathrel{|\!\blacktriangleright} K$, and assume that $w \in K$. If $z \in H$, then so are x and y. So, suppose $z, w \in K$. If $G = H \blacktriangleright K$, then $x, y \in K$ since they are connected to K via non-$\rightharpoonup$ edges. If $G = H \mathrel{|\!\blacktriangleright} K$, then $y \in K$ and either $x \frown y$ or $x \in K$. $\square$

Lemma 3.16. *G is* wing-free*, where a* wing *is an induced subgraph of the form*

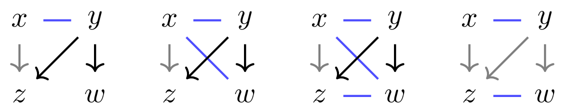


Proof. By induction on the complexity of G. Suppose H and K are MMBI-constructed wing-free graphs, and suppose $G = H \blacktriangleright K$ or $G = H \mathrel{|\!\blacktriangleright} K$ and that the above graph appears as a subgraph of G. As in the proof of the previous lemma, it is safe to assume $w \in K$. There are several possibilities to consider.

1. If $z \in H$, then either $y \in K$ and $z \frown y$, or $y \rightharpoonup w$.
2. If $x \in H$, then either $z \in K$ and $x \frown z$, or $z \in H$.
3. if $y \in H$, then either $x \in H$ and $x \rightharpoonup w$, or $x \in K$ and there is no edge $x \frown y$.

In any case, the above graph does not appear as an induced subgraph of G. □

Lemma 3.17. *G is* trestle-free*, where a* trestle *is an induced subgraph of one of the forms*



(i) (ii) (iii) (iv)

Proof. By induction on the complexity of G. Let H and K be MMBI-constructed trestle-free graphs, and assume G is one of $H \blacktriangleright K, H \mid\!\blacktriangleright K$. Suppose on of (i)-(iv) appears in G. There are several possibilities to consider.

1. If $x \in H$ and $z, y, w \in K$, then z is a root of K and w is not. The vertices y, z, w, v induce an L-shaped graph in K, for any $v \in (w \rightharpoonup)$.
2. In (i)-(iii), if $y \in H$ and $x, y, w \in K$, then there mustn't be an edge $x \frown y$. In (iv), x, z, w induce a dangling root in K.
3. If $x, y \in H$ and $z, w \in H$, then $x \rightharpoonup w$ in G.

□

The following is a reproduction of the the key lemma from [2] needed to characterise arenas by L-freeness and Σ-freeness.

Proposition 3.18. *Let $G = (G, \rightharpoonup)$ be any dag.*

*(i) If G is L-free, then G is **stratified**. That is, if $v \rightharpoonup w$, then* $\mathrm{depth}(v) = \mathrm{depth}(w) + 1$.

*(ii) If G is L-free, then the cones of G are **full**. That is, if $v \rightharpoonup^n w$ and $v \rightharpoonup^{n+1} z$ in G, then $w \rightharpoonup z$.*

(iii) If G is L-free and Σ-free, then $v \rightharpoonup^n y$ and $w \rightharpoonup^m y$ implies either $(v \rightharpoonup^n) \subseteq (w \rightharpoonup^m)$ or $(w \rightharpoonup^m) \subseteq (v \rightharpoonup^n)$.

In the more colourful setting of $(1, 2)$-mixed dags, more can be said regarding (i) and (ii).

Lemma 3.19. *In a $(1, 2)$-mixed dag G that is L-free and Σ-free, split-variation-free, and join-variation-free, both of the following hold.*

(i) If $v \rightharpoonup^n w$ and $v \rightharpoonup^n x \rightharpoonup z$, then $w \rightharpoonup z$.

(ii) If $v \rightharpoonup^n w$ and $v \rightharpoonup^n x \rightharpoonup z$, then $w \rightharpoonup z$.

This essentially means that the levelsets of a cone are monochromatic.

Proof. The case $n = 0$ is trivial. Suppose that we are in situation (i), that for some $n > 0$ we find $v \rightharpoonup^n w$ and $v \rightharpoonup^n x \rightharpoonup z$ in G. By the previous proposition, cones are full in G. This puts $w \rightharpoonup z$ in G, whose colour is to be determined, as well as $u \rightharpoonup w$, $u \rightharpoonup x$ for some $u \in (v \rightharpoonup^{n-1})$. This forces $w \rightharpoonup z$ by the join-variation-free property of MMBI-constructed dags. A similar argument applies to the case of situation (ii). □

4 The First Reconstruction Theorem

In this section, $\sim_{\mathscr{G}}$-equivalence classes of MMBI-formulas are recovered from their MMBI-constructed graphs. The basic outline of how to recover φ from $\mathscr{G}(\varphi)$ is an associated simple graph called its *blueprint*. To see ow these are used, a short detour is in order.

Blueprints

In [2], formulas are recovered from their respective prearenas using mainly part (ii) of proposition 3.18. This property implies that if a prearena G is root connected, then there is a leaf $v \in G$ whose cone $\mathrm{Cone}(v)$ contains all of the roots of G (such a cone is said to be *maximal*). Where $n = \mathrm{depth}(v)$, let H be the subgraph of G induced by the set of vertices

$$Y = \{w \mid \mathrm{Cone}(w) \cap (v\rightharpoondown^{n-1}) \neq \varnothing\},$$

and let K be the rest of G. Then, simply, $G = H \blacktriangleright K$.

Of course, while the underlying dags of BI-constructed graphs are prearenas, there could be many distinct sets of "pre-roots" $Y_1, \ldots, Y_n$, giving distinct "antecedent graphs" $H_1, \ldots, H_m$. Moreover, these graphs could be connected to the "consequent graph" $K = G - (H_1 + \cdots + H_m)$ with a variety of differently coloured arrows, and could be interconnected by webs of undirected edges. While this makes recovering a formula from a BI-constructed graph more complicated, there is a simple way to organize the relevant information.

Let G be a $(1, 2)$-mixed graph such that

(a) G is root connected,

(b) G is L-free and Σ-free,

(c) G is split-variation-free and join-variation-free, and

(d) G has full web connections.

Then G admits at least one maximal cone $\mathrm{Cone}(v)$ containing all of the roots of G. For any u, let

$$Y_u = \{w \mid \mathrm{Cone}(w) \cap (u\rightharpoondown^{\mathrm{depth}(u)-1}) \neq \varnothing\},$$

and denote by $Y_1, \ldots, Y_m$ the distinct Y_u in G for which $\mathrm{Cone}(u)$ is a maximal cone containing the roots of G. For each i, let H_i be the *component* subgraph of G induced by Y_i, and K be the *remainder* graph, $G - (H_1 + \cdots + H_m)$. By (c), each H_i is connected to K with at most one colour of arrow, and label the H_i accordingly: If H_i is connected to K with magic arrows, write $H_i^{\bullet}$, and write $H_i^{\bullet}$ otherwise.

Definition 4.1. The *blueprint* $\mathcal{B}(G)$ of G is the decorated simple graph defined as follows:

$$V(\mathcal{B}(G)) = \{H_i^{\circ} \mid i = 1 \ldots m\} \cup \{K \mid K \backslash \sqrt{G} \neq \varnothing\}$$
$$E(\mathcal{B}(G)) = \{X \frown Y \mid \exists v \in X. \exists w \in Y. v \frown w \in G\}.$$

The *restricted blueprint* is defined $\mathcal{B}^*(G) = \mathcal{B}(G) - K$.

Note that the colour labelling of each vertex is included in the definition, that K is the only uncoloured vertex, and that K is only present as a vertex if K contains some non-root of G. Note also that $\mathcal{B}^*(G) \neq \varnothing$ if and only if G has positive depth (ie. G has more than one vertex, since G is root connected).

The first property that $\mathcal{B}(G)$ inherets from G is N-freeness.

Lemma 4.2. *If G is N-free, then $\mathcal{B}^*(G)$ is N-free.*

Proof. See lemma 3.6. Essentially, $\mathcal{B}^*(G)$ is a subgraph of $\mathcal{K}_1(G)$. □

Definition 4.3. For any subgraph D of the blueprint $\mathcal{B}(G)$ with vertices $\{H_i^\circ \mid i \in I\}$ such that G satisfies (a)-(d), define

$$[D] = \sum\{H_i^\circ \mid H_i \in V(D)\}$$

as a subgraph of G.

Simply put, $[D]$ is the subgraph of G induced by the vertices in the components H_i° of G appearing in D. In particular, of course, $G = [\mathcal{B}(G)] + K$.

The following is a useful inductive device used in the reconstruction theorem to follow. It essentially digs out the "absolute consequent" of an MMBI-formula.

Definition 4.4. Let φ be any propositional formula of MMBI. The *remainder* $\mathbf{k}(\varphi)$ of φ is defined as follows: If $\varphi = \psi \Rightarrow \chi$ or $\varphi = \psi \mathrel{-\!\!*} \chi$, then $\mathbf{k}(\varphi) = \mathbf{k}(\chi)$. Otherwise, $\mathbf{k}(\varphi) = \varphi$.

For example, $\mathbf{k}(p \Rightarrow (a \wedge q)) = a \wedge q$, and

$$\mathbf{k}(a \mathrel{-\!\!*} (b \Rightarrow (c \mathrel{-\!\!*} (d \Rightarrow e)))) = e.$$

The following lemma reveals that the remainders are invariant under $\mathscr{G}$. An important concept used in its proof is that of the *remainder depth* $d_{\mathbf{k}}(\varphi)$ of an MMBI-formula φ. It is defined recursively as follows: If $\varphi \in \{p, \psi \wedge \chi, \psi * \chi\}$, then $d_{\mathbf{k}}(\varphi) = 0$. Otherwise, if $\varphi = \psi \Rightarrow \chi$ or $\varphi = \psi \mathrel{-\!\!*} \chi$, then $d_{\mathbf{k}}(\varphi) = d_{\mathbf{k}}(\chi) + 1$.

Lemma 4.5. *Let $G = \mathscr{G}(\varphi)$ for some formula φ, and denote the remainder graph of G, $G - \sum_i H_i$, by K. Then*

$$\mathscr{G}(\mathbf{k}(\varphi)) = K.$$

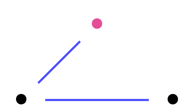
Proof. We proceed by induction on the remainder depth of φ. If $\varphi = p$, then $K = G$ because $\mathcal{B}(G)$ has no components (see the definition). If $\varphi = \psi * \chi$ or $\varphi = \psi \wedge \chi$, then $\mathbf{k}(\varphi) = \varphi$. In such a case, there are no component graphs in the blueprint of G, because a BI-constructed graph has a cone that contains its roots if and only if it is *not* of one of the forms $\mathscr{G}(\psi * \chi)$ or $\mathscr{G}(\psi \wedge \chi)$. Whence, $K = \mathscr{G}(\varphi) = G$.

Instead, assume $\varphi = \psi \Rightarrow \chi$ or $\varphi = \psi \mathrel{-\!\!*} \chi$. Since $\mathbf{k}(\varphi) = \mathbf{k}(\chi)$, it suffices to show that $K = \mathbf{k}(\chi)$. This follows from the observation that every root of the induced subgraph $\mathscr{G}(\psi)$ is connected to every root of G, and therefore must be contained in the (union of the) component graphs of G. □

Structural Characteristics of Blueprints

It is now time to use the results of the previous section to study the structural properties of $\mathcal{B}(G)$.

Lemma 4.6. *If G satisfies conditions (a)-(d) of section 4, and is additionally slice-variation-free, then $\mathcal{B}^*(G)$ is free of the following induced subgraph*



This graph will also be referred to as a slice-variation *in $\mathcal{B}(G)$.*

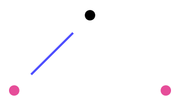
Proof. Suppose $H_1^\bullet \frown H_2^\bullet \frown H_3^\bullet$ appears in $\mathcal{B}(G)$. By full web connectedness of G, there are vertices $v_i \in H_i$ such that $v_1 \frown v_2 \frown v_3$ in G. However, given some root r of K, $v_1 \rightharpoonup r$ while $v_2 \rightharpoonup r$ and $v_3 \rightharpoonup r$. Whence, v_1, v_2, v_3, r induce a slice-variation in G. □

This basically means that, if D is a connected component of the normal slice of $\mathcal{B}(G)$ and $H_i^\bullet$ is a magic component of G such that $H_i^\bullet \frown H_j^\bullet$ for some $H_i^\bullet \in D$, then

$$\forall H_j^\bullet \in D. H_i^\bullet \frown H_j^\bullet.$$

A little more can be said in this direction.

Lemma 4.7. *If G satisfies (a)-(d) of section 4, and is additionally wing-free, then $\mathcal{B}^*(G)$ is free of the following induced subgraph.*



This graph will also be referred to as a wing *in $\mathcal{B}(G)$.*

Proof. Similar to lemma 4.6, but with wings instead of slice-variations. □

This means that there is at most one connected component of $\mathcal{B}^*(G)$ that contains both normal and magic components of G. Combining lemma 4.6 and lemma 4.7, if G satisfies (**a**)-(**d**) and G is slice-variation-free and wing-free, $\mathcal{B}^*(G)$ is of the form

$$D_1 \sqcup \cdots \sqcup D_n \sqcup E$$

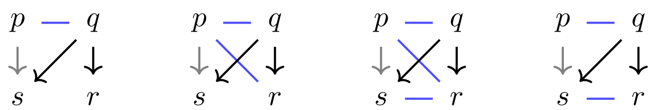
where each D_i is a connected component of normal vertices, and E is a connected graph of either just magic vertices, or of both normal and magic vertices. Together with the following lemma, this observation leads to a limited structural characterisation of MMBI-constructed graphs.

Lemma 4.8. *Assume G satisfies (a)-(d) of section 4, is slice-variation-free and wing-free, and that $K - \sqrt{G} \neq \varnothing$. If, in addition, G is trestle-free and free of dangling sources, then $\mathcal{B}(G)$ is free of the following two subgraphs.*

(*i*) (*ii*)

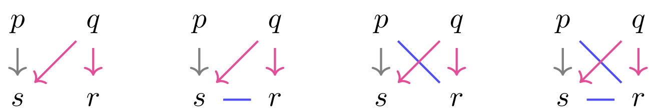
Proof. Let $p \in K - \sqrt{G}$ be a vertex of depth 1, with $p \rightharpoonup s$ and $s \in \sqrt{G}$. Since p is in the remainder of G, there is a root $r \in \sqrt{G} - \mathrm{Cone}(p)$.

To see that the graph (i) cannot appear in $\mathcal{B}(G)$, simply observe that such a normal component of G would imply there being a vertex q, two normal edges $q \rightharpoonup s$ and $q \rightharpoonup r$, and an edge $p \frown q$ in G. In other words, p, q, r, s induce a subgraph of one of the following forms.



In other words, the vertices p, q, s, r form a trestle.

To see that (ii) cannot appear in G, observe that the presence of such a magic component would imply there being a vertex q, two edges $q \rightharpoonup s$ and $q \rightharpoonup r$. Thus, p, q, r, s induce one of the following subgraphs of G.



In other words, p, q, r, s induce a dangling source in G. $\square$

The lemma essentially says that introducing the vertex K into the blueprint $\mathcal{B}(G)$ of G is superfluous. In other words, $\mathcal{B}^*$ and $\mathcal{B}$ carry the same information for MMBI-constructed graphs. The following proposition provides a somewhat restricted characterization of MMBI-constructed graphs.

Proposition 4.9. *Suppose G satisfies (a)-(d) of section 4, is slice-variation-free and wing-free, is trestle-free and free of dangling sources, and is ladder-free as well. then*

$$\mathcal{B}^*(G) = D_1 \sqcup \cdots \sqcup D_n \sqcup E,$$

where each D_i is a connected component of normal vertices, and E is a connected component of either just magic vertices, or of both normal and magic vertices. If E contains only magic edges, there are two possibilities:

(i) If E is empty, then

$$G = ([D_1] \sqcup \cdots \sqcup [D_n]) \blacktriangleright K.$$

(ii) If E consists of only magic vertices, then

$$G = ([D_1] \sqcup \cdots \sqcup [D_n]) \blacktriangleright ([E] \mathbin{|\!\blacktriangleright} K).$$

Proof. The first part of the proposition follows from lemma 4.6 and lemma 4.7. The second part of the proposition is slightly trickier, as there is no reason *a priori* to believe that all of the $\frown$ edges in G are created in either the construction in (i) or the construction in (ii). The vertices are, on the other hand, all created, as

$$G = [D_1] + \cdots + [D_n] + [E] + K.$$

It follows from ladder-freeness that there are no $\frown$ edges between vertices of any $[D_i]$ and K, or between vertices of $[E]$ and K. Hence, the only $\frown$ edges in G that are not induced by roots of G occur either between two vertices in some $[D_i]$, between two vertices in $[E]$, between two vertices in K, or between some vertex of $[E]$ and some vertex of $K - \sqrt{K}$. These are precisely the $\frown$ edges created in the construction of

$$([D_1] \sqcup \cdots \sqcup [D_n]) \blacktriangleright ([E] \mathbin{|\!\blacktriangleright} K).$$

This ends the proof. $\square$

Define, for the sake of brevity,

$$D = D_1 \sqcup \cdots \sqcup D_n.$$

The above picture of MMBI-constructed graphs is simple, but it makes a hefty assumption about the structure of E. A more careful analysis of the structure of E is needed when E consists of both normal and magic components.

Lemma 4.10. *With the same assumptions on G and the same representation of $\mathcal{B}^*(G)$ as in proposition 4.9, assume $S, M \neq \varnothing$, and let S be the normal slice of E and M be the magic slice of E. The following statements hold.*

*(**a**) There is a nonempty subgraph $M' \subseteq M$ such that $p \frown v$ for any $p \in M'$ and any $v \in E - M'$.*

*(**b**) There are $M_1, \ldots, M_m \subseteq M$ and $S_1, \ldots, S_m \subseteq S$ such that either*

$$[E] + K = [M_1] \mathbin{|\!\blacktriangleright} ([S_1] \blacktriangleright (\cdots ([M_m] \mathbin{|\!\blacktriangleright} ([S_m] \blacktriangleright K)) \cdots))$$

or

$$[E] + K = [M_1] \mathbin{|\!\blacktriangleright} ([S_1] \blacktriangleright (\cdots ([S_{m-1}] \mathbin{|\!\blacktriangleright} ([M_m] \mathbin{|\!\blacktriangleright} K)) \cdots))$$

if $S_m = \varnothing$.

*(**c**) The graph G is of one of the following forms.*

Form #	$\varnothing = \ldots$	*Form of G*
I	*none*	$[D] \blacktriangleright ([M_1] \mid\!\blacktriangleright ([S_1] \blacktriangleright (\cdots ([M_m] \mid\!\blacktriangleright ([S_m] \blacktriangleright K)) \cdots)))$
II	D	$[M_1] \mid\!\blacktriangleright ([S_1] \blacktriangleright (\cdots ([M_m] \mid\!\blacktriangleright ([S_m] \blacktriangleright K)) \cdots))$
III	D, S_m	$[M_1] \mid\!\blacktriangleright ([S_1] \blacktriangleright (\cdots ([S_{m-1}] \blacktriangleright ([M_m] \mid\!\blacktriangleright K)) \cdots))$
IV	S_m	$[D] \blacktriangleright ([M_1] \mid\!\blacktriangleright ([S_1] \blacktriangleright (\cdots ([S_{m-1}] \blacktriangleright ([M_m] \mid\!\blacktriangleright K)) \cdots)))$
V	M_i, S_i	$[D] \blacktriangleright K$
VI	D, E	*a single vertex/not root connected*

*Proof of (**a**).* This will be done in two parts: First, it will be shown that the set

$$M_1 = \{x \in M \mid \forall v \in S. x \frown v\}$$

is nonempty, and then it will be shown that M_1 satisfies the desired property.

Let $S_1, \ldots, S_m$ be the connected components of S. The proof proceeds with an induction on m. If $m = 1$, then $M = M_1$ by lemma 4.6. Now assume there are nonempty subsets $M'', M''' \subseteq M$ such that M'' enjoys full edge connections with $S_1 \sqcup \cdots \sqcup S_m$ and M'' enjoys full edge connections with $S_2 \sqcup \cdots \sqcup S_{m+1}$. Let $v \in S_1$, $w \in S_{m+1}$, $y \in M''' - M''$, and $x \in M'' - M'''$. Since v and w are in separate S_i, there is no edge $v \frown w$. However, it must be the case that one of $x \frown w$ or $y \frown v$ or $x \frown y$, by lemma 4.6. Appealing to lemma 4.2, however, it must be the case that either $x \frown w$ or $y \frown v$. In the first case, observe that $M'' \subseteq M_1$, and in the second, observe that $M''' \subseteq M_1$. This shows that $M_1 \neq \varnothing$.

To see that M_1 satisfies the desired property, let $y \in E - M_1$ and $x \in M_1$. If $y \in S$, then by assumption $x \frown y$, so assume $y \in M - M_1$. There is a normal vertex $v \in S$ for which there is no edge $y \frown v$. However, $x \frown v$. Thus, by lemma 4.7, $x \frown y$. This shows that M_1 is the desired set M' (and, in fact, is the largest such set).

*Proof of (**b**).* Define $E_0 = E$,

$$M_1 = \{x \in M \mid \forall v \in S. x \frown v\},$$

and set $E_1 = E - M_1$. Since slice-variation-freeness, wing-freeness, and N-freeness are hereditary properties, any induced subgraph of E also has these properties. In particular, the graph E_1 has these properties.

Define S_1 to be the set of normal vertices in E_1 not connected to a magic vertex. The set S_1 is nonempty by construction, since if every vertex in S_1 were connected to a magic vertex in E_1, E_1 would satisfy the conditions in part (**a**) and there would be a magic vertex in E_1 connected to all of S, contradicting the assumption that $E_1 \cap M_1 = \varnothing$. So, let $E_1' = E_1 - S_1$.

Now, if $E_1' = \varnothing$, then $E = M_1 \sqcup S_1$ and

$$[E] + K = [M_1] \mid\!\blacktriangleright ([S_1] \blacktriangleright K).$$

Otherwise, proceed inductively.

Define $E_0' = \varnothing$, and

$$\begin{aligned}
M_{i+1} &= \{x \in M \cap E_i' \mid \forall v \in S \cap E_i'. x \frown v\} \\
E_{i+1} &= E_i' - M_{i+1} \\
S_{i+1} &= \{v \in S \cap E_{i+1} \mid \neg\exists x \in M \cap E_{i+1}. v \frown x\} \\
E_{i+1}' &= E_{i+1} - S_{i+1}.
\end{aligned}$$

Remember that $E_i' = (M \cap E_i') \sqcup (S \cap E_i')$ for all i. There is a smallest m such that $E_m' = \varnothing$, for the following reason: Suppose $E_i' \neq \varnothing$. Since E_i' satisfies the conditions on E from part (**a**), the set M_{i+1} is nonempty, meaning that either $M_{i+1} = E'_i$ or there are normal vertices in E_{i+1}. In the first case, $E'_{i+1} \subseteq E_{i+1} = \varnothing$. In the second, S_{i+1} is nonempty and therefore $E''_{i+1} \subsetneq E_{i+1} \subseteq E'_i$. Such an m exists, now, because E is finite. Computing recursively,

$$\begin{aligned}
E &= E_1 \sqcup M_1 \\
&= (E'_1 \sqcup S_1) \sqcup M_1 \\
&= (E_2 \sqcup M_2) \sqcup S_1 \sqcup M_1 \\
&= (E'_2 \sqcup S_2) \sqcup M_2 \sqcup S_1 \sqcup M_1 \\
&\ \ \vdots \\
&= E'_m \sqcup S_m \sqcup M_{m-1} \sqcup \cdots \sqcup S_1 \sqcup M_1 \\
&= \varnothing \sqcup S_m \sqcup M_{m-1} \sqcup \cdots \sqcup S_1 \sqcup M_1 \\
&= S_m \sqcup M_{m-1} \sqcup \cdots \sqcup S_1 \sqcup M_1.
\end{aligned}$$

In the construction above, observe that since E'_i satisfies the conditions on E in part (**a**), M_{i+1} satisfies the property that for any $x \in M_{i+1}$ and any $y \in E'_i - M_{i+1} = E_{i+1}$, $x \frown y$. Because $S_j, M_{j+1} \subseteq E_{i+1}$ for any $j \geqslant i$, M_i has full edge connections with $S_j \cup M_{j+1}$ whenever $i \leqslant j$. Conversely, by construction, if $x \frown y$ in $[E]$, then either

(i) $x, y \in M$,

(ii) $x, y \in S_i$ for some $i \leqslant m$, or

(iii) there are i, j with $i < j \leqslant m$ such that $x \in M_i$ and $y \in S_j$.

Points (i) and (ii) are clear possibilities. To see why (iii) is the only remaning possibility, suppose $i < j$, and observe that

$$M_j \subseteq E'_j \cap M \subseteq E'_i \cap M \subseteq E_i \cap M.$$

By definition, S_i shares no connections with $E_i \cap M$, leaving possibility (iii). The second part of the lemma now follows from ladder-freeness, as in the proof of proposition 4.9(ii).

*Proof of (**c**).* Immediate from parts (**a**), (**b**), and ladder-freeness, as in the proof of proposition 4.9. □

Arenas and their Characterisization

Definition 4.11. A $(1, 2)$-mixed dag G is a *MMBI-arena* if G

(**a**) is L-free and Σ-free (see lemma 3.2),

(**b**) is N-free (see lemma 4.2) and ladder-free (see lemma 3.10),

(**c**) is split-variation-free (see lemma 3.7) and join-variation-free (see lemma 3.8),

(**d**) is slice-variation-free (see lemma 3.15) and wing-free (see lemma 3.16),

(**e**) is trestle-free (see lemma 3.17) and free of dangling sources (see lemma 3.14),

(**f**) is box-free (see lemma 3.12) and $\boxslash$-free (see lemma 3.13),

(**g**) is free of dangling roots (see lemma 3.9) and trailing roots (see lemma 3.11), and

(**h**) has full web connections (see lemma 3.5).

Lemma 4.12. *Let G be a root connected MMBI-arena with components $H_1, \ldots, H_m$ and remainder K. Then $H_1, \ldots, H_m$ and K are MMBI-arenas.*

Proof. That each H_i and K satisfy (**a**)-(**f**) is immediate, as these are hereditary properties. Furthermore, K

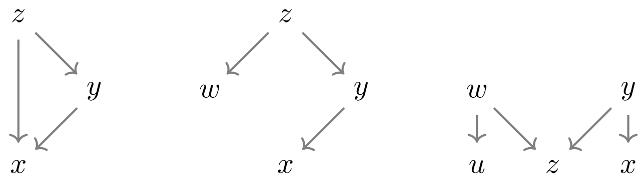
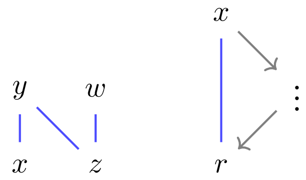
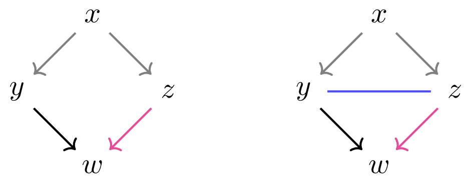
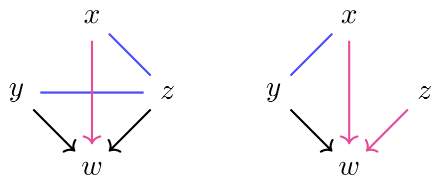
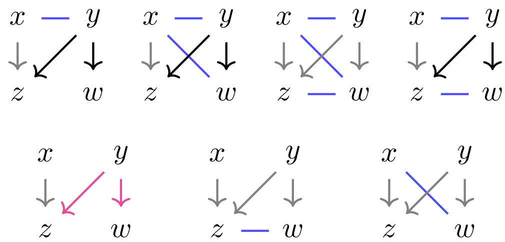
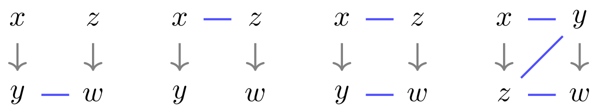
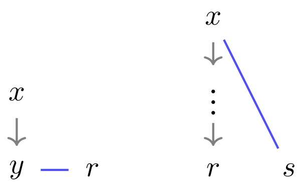
Condition	Forbidden graph
L-free, Σ-free	
N-free, ladder-free	
join-variation-free	
slice-variation-free, wing-free	
trestle-free, free of dangling sources	
box-free, $\boxslash$-free	
free of dangling roots, free of trailing roots	

Table 1: Table of graphs forbidden from MMBI-arenas

satisfies (**g**) because $\sqrt{K} = \sqrt{G}$, and every H_i satisfies (**h**) because root connected components of $H_i - \sqrt[k]{H_i}$ are root connected components of $G - \sqrt[(k+1)]{G}$.

To see that H is free of trailing roots, let

$$s \frown x \rightharpoonup x_1 \rightharpoonup \cdots \rightharpoonup x_n = r$$

be a trailing root in H. Since s is a root of H, there is a root y of G and an edge $s \rightharpoonup y$. If there are no additional edges in H, the vertices x, x_1, s, y induce a box in G.

To see that the H_i are free of dangling roots, let $H = H_i$ for some i, and suppose $x \rightharpoonup y \frown z$ appears in H

with z a root of H. Since H is root connected, there is a vertex v at depth 1 such that $(v\rightharpoonup)$ contains all of the roots of H. Assuming there is no edge between x and z, there are two possibilities.

1. There is no edge $v \frown x$. It is not possible for $y \in \sqrt{G}$, as this would make x a dangling source in the graph induced by x, v, y, z. However, if there is no edge $v \rightharpoonup y$, x, y, v, z induce a box.
2. There is an edge $v \frown x$. Either $v \rightharpoonup y$ or $v \frown y$. If $v \rightharpoonup y$, then x, y, v, z induce a trestle. If $v \frown y$, then x, y, z, w form a $\boxslash$-shaped graph.

This shows that H satisfies (**g**).

To see that K satisfies (**h**), first observe that since root connected components of $K - \sqrt{K}$ are root connected components of $G - \sqrt{G}$, $K - \sqrt{K}$ has full web connections. This leaves the following possibilities: Let K_1 and K_2 be a pair of root connected components of K. There are a few cases to consider.

1. Suppose there is an edge $r_1 \frown r_2$ for some $r_1 \in \sqrt{K_1}$ and $r_2 \in \sqrt{K_2}$. It suffices to show that, if $\mathrm{Cone}(v)$ is a cone containing r_1 and $\mathrm{Cone}(w)$ is a cone containing r_2, then $\mathrm{Cone}(v)$ and $\mathrm{Cone}(w)$ enjoy full web connections. Towards this end, let $y \rightharpoonup^* r_2$ and
$$y \rightharpoonup y_1 \rightharpoonup \cdots \rightharpoonup y_m \rightharpoonup r_2.$$
It cannot be the case that there is no edge $r_1 \frown y_m$, for otherwise r_1, y_m, r_2 would induce a dangling root. A simple induction reveals that $r_1 \frown y$ and $r_1 \frown y_i$ for $i = 1, \ldots, m$. Similarly, if
$$x \rightharpoonup x_1 \rightharpoonup \cdots \rightharpoonup x_n \rightharpoonup r_1,$$
then $x \frown r_2$ and $x_i \frown r_2$ for each $i = 1, \ldots, n$. Moreover, $x_n \frown y_m$, for otherwise x_n, r_1, r_2, y_m would induce a box. Since $K - \sqrt{K}$ has full web connections, $x \frown y$ and $x_i \frown y_j$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

   To see that $\mathrm{Cone}(v) \frown \mathrm{Cone}(w)$, it now suffices to show that $r_1 \frown r$ for any other root r of $\mathrm{Cone}(w)$, and $s \frown r_2$ for any other root s of $\mathrm{Cone}(v)$. By symmetry, only the former requires argument. Since r and r_2 are roots in the same connected component, there is a $t \in K_2$ such that $t \rightharpoonup r_2$ and $t \rightharpoonup r$. By the previous observation, $r_1 \frown w$. If there is no edge $r_1 \frown r$, then r_1, t, r induce dangling roots. Hence, $r_1 \frown r$, which concludes this case.
2. Let $y \in K_2$ be a non-root, and suppose that $r_1 \frown y$. Let
$$y \rightharpoonup y_1 \rightharpoonup \cdots \rightharpoonup y_m \rightharpoonup r$$
be any path from y to a root of K_2. If $m = 0$, then it must be the case that $r_1 \frown r$, else r_1, r, y form a trailing root in K. If $m > 0$, let k be the largest index for which there is no edge $r_1 \frown y_k$. Again, if there is no edge $r_1 \frown r$, then $r_1, y_k, y_{k+1}, \ldots, y_m, r$ form a trailing root in K. Whence, $r_1 \frown r$, returning to Case 1.
3. There are non-roots $x \in K_1$ and $y \in K_2$ such that $x \frown y$. It suffices to show that, for some root $r_1 \in K_1$, $r_1 \frown y$. Since $K - \sqrt{K}$ has full web connections, one can assume without loss of generality that x and y are at depth 1: For some roots $r_1 \in K_1$ and $r_2 \in K_2$, $x \rightharpoonup r_1$ and $y \rightharpoonup r_2$. If there are no edges other than those explicitly mentioned, then x, y, r_1, r_2 induce a box-shape in G.

This shows that K satisfies (**h**). □

Lemma 4.13. *Let G be a MMBI-arena. If D is a root connected component of G, then D is a MMBI-arena.*

Proof. Since they are hereditary properties, it is clear that D satisfies (**a**)-(**f**). As a root connected component of $D - \sqrt[m]{D}$ is, in particular, a root connected component of $G - \sqrt[(m+n)]{G}$, D satisfies (h) by definition. To see that D is free of dangling or trailing roots, observe that every dangling or trailing root in D is also a dangling or trailing root in G, because $\sqrt{D} \subseteq \sqrt{G}$. $\square$

We are now ready for the main theorem of the whole document.

Theorem 4.14. *Let G be a $(1,2)$-mixed graph. Then G is a* MMBI*-arena if and only if G is* MMBI*-constructed.*

Proof. We have already seen that every MMBI-constructed graph is a MMBI-arena. To see the forward direction, proceed by induction on the size of G. If $|G| = 1$, there is nothing to see. So, assume $|G| > 1$, and let $D_1, \ldots, D_m$ be the root connected components of G. If $m > 0$, each of the D_i is a MMBI-constructed $(1,2)$-mixed graph by hypothesis and by lemma 4.13. Furthermore, by lemma 3.6 and the equivalence between N-free simple graphs and cographs, there is a $\{*, \wedge\}$-formula $\varphi(D_1, \ldots, D_m)$ detailing the construction of $\mathcal{K}(G)$ from the "propositional variables" $D_1, \ldots, D_m$. Where $D_i = \mathscr{G}(\psi_i)$, then

$$G = \mathscr{G}(\varphi(\psi_1, \ldots, \psi_m)).$$

If $m = 1$, a different approach must be taken. Consider the restricted blueprint $\mathcal{B}^*(G)$ of G, let $H_1, \ldots, H_l$ be the component vertices in $\mathcal{B}^*(G)$, and let $K = G - \sum_i H_i$ be the remainder of G. Since G is root connected, there is at least one non-empty component vertex H_i° in the blueprint of G. By lemma 4.10, $\mathcal{B}(G)$ is of one of the forms I-V. Borrow for the time being the notation used in lemma 4.10. Since lemma 4.12 and the inductive hypothesis state that $H_1, \ldots, H_l$, and K are MMBI-constructed, there are MMBI formulas ψ_i, θ_i, δ, and χ such that

$$[M_i] = \mathscr{G}(\psi_i), \; [S_i] = \mathscr{G}(\theta_i), \; [D] = \mathscr{G}(\delta), \quad \text{and} \quad K = \mathscr{G}(\chi).$$

The ψ_i and θ_i are constructed from the magic and normal components in E using the fact that $[E]$ is a disjoint union of MMBI-constructed graphs, and similarly δ is constructed from the components in D.

The hardest case to consider next is when G has form #I. All other form #s are handled similarly. If G has form #I, let

$$\gamma_G = \delta \Rightarrow (\psi_1 \twoheadrightarrow (\theta_1 \Rightarrow (\cdots(\psi_m \twoheadrightarrow (\theta_m \Rightarrow \chi))\cdots))).$$

Then $G = \mathscr{G}(\gamma_G)$ by construction. $\square$

The implicit role played by **P**-labels in the proof of the previous theorem deserves some attention. If a $(1,2)$-mixed dag G comes with a **P**-labelling, and also happens to be an MMBI-arena, then one can ensure that the proposition letters appearing in the formulas constructed to represent the various portions of the underlying graph of G coincide with the **P**-labels of their corresponding vertices in G. This is simply because there is no specification of the **P**-labelling inherent in the graph operations: Simply use any proposition letters you want during formula construction, then replace them with the labels of their corresponding vertices in G! The proposition occurrences in γ_G are in one-to-one correspondence with the vertices in $\mathscr{G}(\gamma_G)$. This observation is also used implicitly in the following theorem.

Theorem 4.15. *Let $G = \mathscr{G}(\varphi)$ be a graph constructed from the MMBI formula φ. Define γ_G to be the formula described in the proof of theorem 4.14, built to satisfy $G = \mathscr{G}(\gamma_G)$. Then $\varphi \sim \gamma_G$*

Proof. By induction on the length of φ. In the case where $\varphi = p$ for some propositional variable p, $G = (\{p\}, \varnothing)$, in which case $\gamma_G = p$. From now on, η and θ will be MMBI formulas shorter than φ, and $H = \mathscr{G}(\eta)$ and $K = \mathscr{G}(\theta)$.

If $\varphi = \eta \wedge \theta$ or $\varphi = \eta * \theta$, G is not root connected. In the proof of theorem 4.14, the graph $\mathcal{K}(G)$ is the simple graph whose vertices are the root connected components of G. Let $D_1, \ldots, D_m$ be the root connected components of H and $D_{m+1}, \dots, D_n$ be the root connected components of K. If $G = H \sqcup K$, then

$$\mathcal{K}(G) = \mathcal{K}(G){\restriction}(D_1, \dots, D_m) \sqcup \mathcal{K}(G){\restriction}(D_{m+1}, \dots, D_n)$$

is the disjoint union of its subgraphs induced by the vertices representing root connected components of H and K seperately. Whence, if $\varphi(D_1, \dots, D_n), \varphi^H(D_1, \dots, D_m)$, and $\varphi^K(D_{m+1}, \dots, D_n)$ are the $\{*, \wedge\}$-formulas detailing the construction of $\mathcal{K}(G), \mathcal{K}(G){\restriction}(D_1, \dots, D_m)$, and $\mathcal{K}(G){\restriction}(D_{m+1}, \dots, D_n)$ respectively, then

$$\gamma_G = \varphi(\gamma_{D_1}, \dots, \gamma_{D_n}) = \varphi^H(\gamma_{D_1}, \dots, \gamma_{D_m}) \wedge \varphi^K(\gamma_{D_{m+1}}, \dots, \gamma_{D_n}).$$

By the inductive hypothesis,

$$\eta \sim \gamma_H = \varphi^H(\gamma_{D_1}, \dots, \gamma_{D_m}) \ \text{ and } \ \theta \sim \gamma_K = \varphi^K(\gamma_{D_{m+1}}, \dots, \gamma_{D_n}).$$

Now, because $\sim$ is a congruence relation,

$$\varphi = \eta \wedge \theta \sim \gamma_H \wedge \gamma_K = \gamma_G.$$

The case in which $\varphi = \eta * \theta$ is similar.

If $\varphi = \eta \Rightarrow \theta$ or $\varphi = \eta \mathrel{-\!\!*} \theta$, then G is root connected, in which case G has a form #I-#V. We now perform a subinduction on the remainder depth $d_{\mathbf{k}}(\varphi)$ (see lemma 4.5) of φ, beginning with the case in which $d_{\mathbf{k}}(\varphi) = 1$. By lemma 4.5, the remainder K of G satisfies

$$K = \mathscr{G}(\mathbf{k}(\varphi)) = \mathscr{G}(\mathbf{k}(\theta)) = \mathscr{G}(\theta).$$

This means $\gamma_K \sim \theta$ by the induction hypothesis. Now, since G has remainder depth 1, either G is of form #III with $m = 1$ or #V. In either case,

$$\mathscr{G}(\psi) = \mathscr{G}(\varphi) - \mathscr{G}(\theta) = \mathscr{G}(\varphi) - K = \sum_i [H_i^\circ],$$

where $H_1^\circ, \dots, H_l^\circ$ are the vertices of $\mathcal{B}^*(G)$. Now, since $\mathscr{G}^*(G)$ is N-free, there is a $\{*, \wedge\}$-formula $\Gamma(H_1^\circ, \dots, H_l^\circ)$ detailing the construction of $\mathcal{B}^*(G)$. It follows from $d_{\mathbf{k}}(G) = 1$ that either $H_i^{\bullet}$ for all i, or $H_i^{\bullet}$ for all i. Hence,

$$\mathscr{G}(\psi) = \sum_i [H_i^\circ] = \mathscr{G}(\Gamma(\gamma_{[H_1^\circ]}), \dots, \gamma_{[H_l^\circ]})).$$

By the induction hypothesis,

$$\psi \sim \Gamma(\gamma_{[H_1^\circ]}), \dots, \gamma_{[H_l^\circ]}),$$

from which it now follows that

$$\varphi \sim \psi \Rightarrow \theta \sim \Gamma(\gamma_{[H_1^\circ]}), \dots, \gamma_{[H_l^\circ]}) \Rightarrow \gamma_K \sim \gamma_G$$

or

$$\varphi \sim \psi \mathrel{-\!\!*} \theta \sim \Gamma(\gamma_{[H_1^\circ]}), \dots, \gamma_{[H_l^\circ]}) \mathrel{-\!\!*} \gamma_K \sim \gamma_G.$$

This concludes the case where $d_{\mathbf{k}}(\varphi) = 1$.

Now assume that the theorem holds for formulas strictly shorter than φ, as well as for formulas which are the same length as φ but have a remainder depth strictly smaller than $d_{\mathbf{k}}(\varphi)$. There are two cases to consider:

- In the first, either $\varphi = \psi \Rightarrow (\theta_1 \Rightarrow \theta_2)$ or $\varphi = \psi \mathrel{-\!\!*} (\theta_1 \mathrel{-\!\!*} \theta_2)$. Here, either

  $$\varphi \sim \varphi' := (\psi \wedge \theta_1) \Rightarrow \theta_2$$

  or

  $$\varphi \sim \varphi' := (\psi * \theta_1) \mathrel{-\!\!*} \theta_2.$$

  In either case, $d_{\mathbf{k}}(\varphi') < d_{\mathbf{k}}(\varphi)$. By the subinduction hypothesis, $\varphi \sim \varphi' \sim \gamma_G$.

- In the second case, either $\varphi = \psi \Rightarrow (\theta_1 \mathrel{-\!\!*} \theta_2)$ or $\varphi = \psi \mathrel{-\!\!*} (\theta_1 \Rightarrow \theta_2)$. If $\varphi = \psi \Rightarrow (\theta_1 \mathrel{-\!\!*} \theta_2)$, then $G = \mathscr{G}(\psi) \blacktriangleright (\mathscr{G}(\theta_1) \mid\!\blacktriangleright \mathscr{G}(\theta_2))$. This makes $[D] = \mathscr{G}(\psi)$ and $[E] + K = \mathscr{G}(\theta_1) \mid\!\blacktriangleright \mathscr{G}(\theta_2)$ in the notation of lemma 4.10, since $\mathscr{G}(\psi)$ appears in $\mathcal{B}^*(G)$ as the set of normal components disconnected from any magic vertex. Given the induction hypothesis, it follows that

$$\gamma_G = \gamma_{[D]} \Rightarrow \gamma_{([E]+K)} \sim \psi \Rightarrow (\theta_1 \mathrel{-\!\!*} \theta_2) = \varphi.$$

  Similarly, if $\varphi = \psi \mathrel{-\!\!*} (\theta_1 \Rightarrow \theta_2)$, then $\mathscr{G}(\psi)$ appears in $\mathcal{B}^*(G)$ as the set of magic components connected to every outside component, normal or magic. This is precisely M_1 in the notation of lemma 4.10. Given the induction hypothesis, it follows that

$$\gamma_G = \gamma_{[M_1]} \mathrel{-\!\!*} \gamma_{([E]+K)} \sim \psi \mathrel{-\!\!*} (\theta_1 \Rightarrow \theta_2) = \varphi.$$

This finishes the proof. □

Corollary 4.16. $\sim_{\mathscr{G}} = \sim$

Proof. We have already seen that $\sim \subseteq \sim_{\mathscr{G}}$ in lemma 3.1. To see the reverse inclusion, let $\mathscr{G}(\varphi) = \mathscr{G}(\psi) = G$. By theorem 4.15,

$$\varphi \sim \gamma_G \sim \psi.$$

Hence, $\varphi \sim \psi$. □

Of course, the equivalence relation $\sim$ encodes an important set of logical equivalences in MMBI. The main results of this section essentially state that the function $\mathscr{G}$ is an example of what I call a *graph constructor* for the logic of bunched implications.

Definition 4.17. Given a logic L and a class of decorated graphs $\mathcal{G}$, a *graph constructor* for L is a function $\mathscr{G} : \mathsf{L} \to \mathcal{G}$ such that if $\mathscr{G}(\varphi) = \mathscr{G}(\psi)$, then the logical equivalence $\varphi \equiv_{\mathsf{L}} \psi$ holds in L.

5 With More Colours

This section is a work in progress, and ends abruptly when the material ends. I do believe, however, that the work here can be carried out without too much pain by somebody who cares enough about building graph constructors for "higher-order" versions of MMBI, and I think pBI is included in this.

The only pBI-connective missing in MMBI-formulas is disjunction, but in fact, a slight modification of $\mathscr{G}_{\mathsf{MMBI}}$ produces a graph constructor that includes this connective. Since the same method applies to a more general class of logics, a general class of graph constructors will be recorded here. Proofs that they do indeed define graph constructors can be found in the next section.

<u>*Remark.*</u> While it is true that only a slight modification of the graph constructor $\mathscr{G}_{\mathsf{MMBI}}$ produces a graph constructor for full pBI, the reason for the ommission is easily stated: A combinatorial proof system for multiplicative intuitionistic logic has already appeared in [2], and the graph constructor given coincides with $\mathscr{G}_{\mathsf{MMBI}}$ on the additive fragment of MMBI. I hopes that the techniques used in [2] extend to a combinatorial proof system for MMBI.

Fix a set I of indices, and fix a pair of binary operators $\otimes_i, \multimap_i$ for each index $i \in I$. Let CC_I be the grammar consisting formulas

$$A, B ::= p \in \mathbf{P} \mid A \vee B \mid A \otimes_i B \mid A \multimap_i B \qquad (i \in I).$$

I assume that the equivalence relation $\equiv_{\mathsf{CC}_I}$ induces a lax commutative semigroup-closed structure $\otimes_i, \multimap_i$ for each $i \in I$. This is to say that the following equations hold

$$
\begin{aligned}
(\varphi \otimes_i \psi) \otimes_i \chi &\equiv_{\mathsf{CC}_I} \varphi \otimes_i (\psi \otimes_i \chi) \\
\varphi \otimes_i \psi &\equiv_{\mathsf{CC}_I} \psi \otimes_i \varphi \\
(\varphi \otimes_i \psi) \multimap_i \chi &\equiv_{\mathsf{CC}_I} \varphi \multimap_i (\psi \multimap_i \chi).
\end{aligned}
$$

Let $\sim_{\mathsf{CC}_I}$ denote the equivalence relation generated by these equations alone.

Denote with $\mathcal{MD}_k$ the class of (k,k)-mixed dags, where $k = |I|$. The relevant graph operations designed to match the connectives of CC_I are defined as follows: Let $H, K \in \mathcal{MD}_k$ with the single colour set I (undirected and directed edges are coloured from the same set), and for each $i \in I$,

$$
\begin{aligned}
H \,||_i\, K &= H \sqcup K + \{v \frown_i w \mid v \in H \text{ and } w \in K\} \\
H \,|\!\blacktriangleright_i\, K &= H \sqcup K + \{r \rightharpoonup_i s \mid r \in v \in \sqrt{H} \text{ and } w \in \sqrt{K}\}
\end{aligned}
$$

where subscripts denote colouring. The relevant mapping on formulas is denoted

$$
\mathscr{G}_k : \mathsf{CC}_I \longrightarrow \mathcal{MD}_k,
$$

and defined recursively as follows: For any $p \in \mathbf{P}$,

$$
\begin{aligned}
V(\mathscr{G}_k(p)) &= \{v^{(p)}\} \\
\frown_i (\mathscr{G}_k(p)) &= \varnothing && (i \in I) \\
\rightharpoonup_i(\mathscr{G}_k(p)) &= \varnothing && (i \in I),
\end{aligned}
$$

and given CC_I-formulas φ and ψ,

$$
\begin{aligned}
\mathscr{G}_k(\varphi \vee \psi) &= \mathscr{G}_k(\varphi) \sqcup \mathscr{G}_k(\psi) \\
\mathscr{G}_k(\varphi \otimes_i \psi) &= \mathscr{G}_k(\varphi) \,||_i\, \mathscr{G}_k(\psi) && (i \in I) \\
\mathscr{G}_k(\varphi \multimap_i \psi) &= \mathscr{G}_k(\varphi) \,|\!\blacktriangleright_i\, \mathscr{G}_k(\psi) && (i \in I).
\end{aligned}
$$

A graph is said to be *CC_I-constructed* if it is of the form $\mathscr{G}_k(\varphi)$ for some CC_I formula φ.

5.1 Basic properties of $\mathscr{G}_k$

Many of the lemmas recorded in section 3 generalize to $\mathscr{G}_k$.

Lemma 5.1. *Define the equivalence relation $\sim_{\mathscr{G}_k}$ so that $\varphi \sim_{\mathscr{G}_k} \psi$ if and only if $\mathscr{G}_k(\varphi) = \mathscr{G}_k(\psi)$, for any CC_I-formulas p and ψ. Then $\sim_{\mathsf{CC}_I} \subseteq \sim_{\mathscr{G}_k}$.*

Proof. Identical to the argument given for lemma 3.1. □

It is also immediate that the underlying directed acyclic graph of a CC_I-constructed graph is a prearena, in the sense of [2]. Thus, proposition 3.18 applies. In fact, a more general version of lemma 3.19 applies as well.

Lemma 5.2. *Let G be a CC_I-constructed graph. The following statements hold.*

1. *G is* split-variation-free, *where a* split-variation *is a pair of directed edges with the same source but of different colours.*

2. G is join-variation-free, *where a* join-variation *is an induced subgraph of the form*

$$\begin{tikzcd}
& x \arrow[dl] \arrow[dr] & \\
y \arrow[dr, "i"] & & z \arrow[dl, "j"'] \\
& w &
\end{tikzcd}$$

in which $i \neq j$.

Proposition 5.3. *In a (k, k)-mixed dag G that is L-free and Σ-free, split-variation-free, and join-variation-free, both of the following holds: If $v \curvearrowright^n w$ and $v \curvearrowright^n x \curvearrowright_i z$, then $w \curvearrowright_i z$.*

The proofs are similar to those of lemma 3.19 and lemma 3.8. Together with the addition of the following lemma, a generalized notion of *blueprint* for the appropriate subset of $\mathcal{MD}_k$ can be given.

Lemma 5.4. *If G is a $\mathcal{CC}_I$-constructed graph, then G has* full i-web connections *for any $i \in I$. That is, the root-connected components of $G - \sqrt[n]{G}$ pairwise enjoy either full $\frown_i$-connections, or no $\frown_i$-connections at all, for any $n \in \mathbb{N}$.*

Fix, for the time being, a $G \in \mathcal{MD}_k$ such that

(a) the underlying dag of G is root-connected,

(b) the underlying dag of G is L-free and Σ-free,

(c) G is split-variation-free and join-variation-free, and

(d) G has full i-web connections for any $i \in I$.

Again, G admits at least one maximal cone containing $\sqrt{G}$. For any $u \in V(G)$, let

$$Y_u = \{w \mid \mathrm{Cone}(w) \cap (u \curvearrowright^{\mathrm{depth}(u)-1} \neq \varnothing\},$$

and let $Y_1, \ldots, Y_m$ be the distinct Y_u for which $\mathrm{Cone}(u)$ is a maximal cone containing $\sqrt{G}$. For each $l = 1 \cdots m$, let H_l denote the *component* subgraph of G induced by Y_l, and adjoint to it the label $H_l^{(i)}$ if i is the colour of the directed edges joining H_l to the roots of G. By proposition 5.3, of course, this labelling is well-defined.

Definition 5.5. The *blueprint* $\mathcal{B}(G)$ of G is the decorated simple graph defined as follows:

$$\begin{aligned}
V(\mathcal{B}(G)) &= \{H_l^{(i_l)} \mid l = 1 \ldots m\} \cup \{K \mid K \backslash \sqrt{G} \neq \varnothing\} \\
E(\mathcal{B}(G)) &= \{X \frown_i Y \mid \exists v \in X . \exists w \in Y . v \frown_i w \in G\}.
\end{aligned}$$

The *restricted blueprint* is defined $\mathcal{B}^*(G) = \mathcal{B}(G) - K$.

As before, $\mathcal{B}(G)$ has coloured edges, as well as coloured vertices except for the *remainder* vertex K, which is only present if there are non-root vertices of K.

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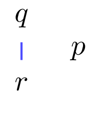
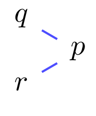
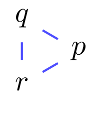
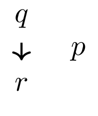
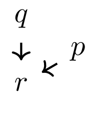
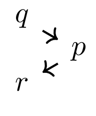
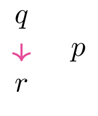
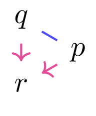
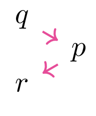
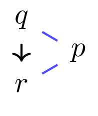
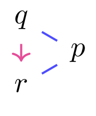
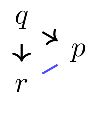
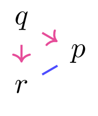
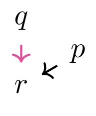
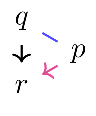
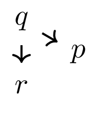
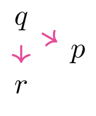
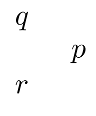
A Examples and Nonexamples of MMBI-arenas

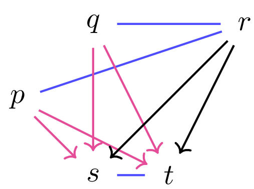
Example A.1. The three-vertex examples are as follows.

φ	$\mathscr{G}(\varphi)$	ψ	$\mathscr{G}(\psi)$	χ	$\mathscr{G}(\chi)$
$p \wedge (q * r)$		$p * (q \wedge r)$		$p * (q * r)$	
$p \wedge (q \Rightarrow r)$		$p \Rightarrow (q \Rightarrow r)$		$(p \Rightarrow q) \Rightarrow r$	
$p \wedge (q \mathrel{-\!*} r)$		$p \mathrel{-\!*} (q \mathrel{-\!*} r)$		$(p \mathrel{-\!*} q) \mathrel{-\!*} r$	
$p * (q \Rightarrow r)$		$p * (q \mathrel{-\!*} r)$		$q \Rightarrow (p * r)$	
$q \mathrel{-\!*} (p * r)$		$p \Rightarrow (q \mathrel{-\!*} r)$		$p \mathrel{-\!*} (q \Rightarrow r)$	
$q \Rightarrow (p \wedge r)$		$q \mathrel{-\!*} (p \wedge r)$		$q \wedge (p \wedge r)$	

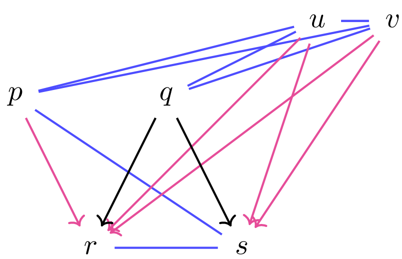
Example A.2. Here is one that uses all four connectives nontrivially: The graph for the formula

$$(p \wedge q) \mathrel{-\!*} (r \Rightarrow (s * t))$$

is

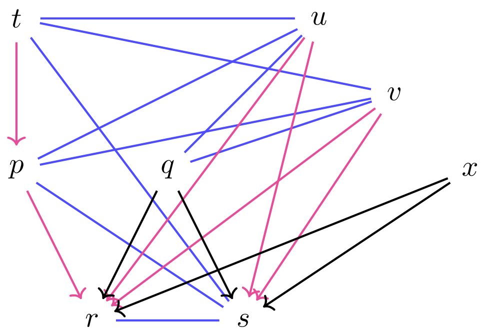


Example A.3. This one is a classic example of an unintuitive BI constructed graph.



This one comes from $(u * v) \mathrel{-\!*} (q \Rightarrow (s * (p \mathrel{-\!*} r)))$. Can you find the induced subgraph that isn't BI constructed?

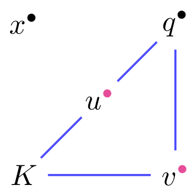
Example A.4. This one has a somewhat nontrivial blueprint.



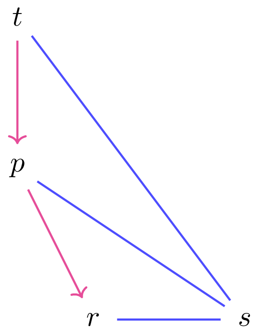
This one comes from the formula

$$x \Rightarrow ((u \wedge v) \mathrel{-\!\!*} (q \Rightarrow (s * ((t \mathrel{-\!\!*} p) \mathrel{-\!\!*} r)))).$$

The blueprint of the above BI constructed graph is



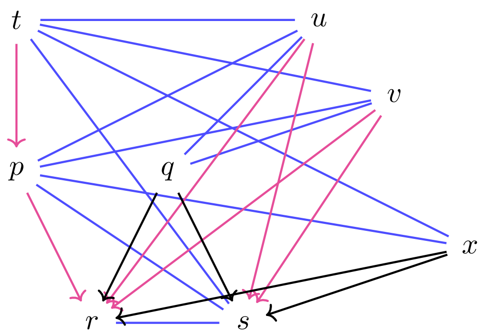
where K is the graph



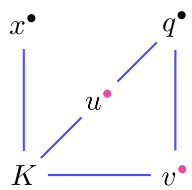
Note that

$$K = \mathscr{G}(s * ((t \mathrel{-\!\!*} p) \mathrel{-\!\!*} r)) = \mathscr{G}\big(\mathbf{k}\big(x \Rightarrow ((u \wedge v) \mathrel{-\!\!*} (q \Rightarrow (s * ((t \mathrel{-\!\!*} p) \mathrel{-\!\!*} r))))\big)\big).$$

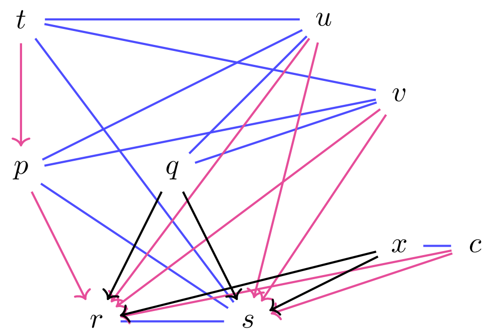
Taking a look at the blueprint, is it clear why the following graph is *not* BI constructed?



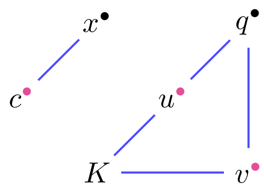
Hint: The blueprint for the above graph is



How about this one?

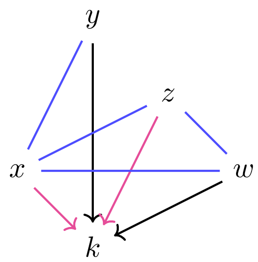


Its blueprint is



The next example is an important one, showing that proposition 4.9 is not enough.

Example A.5. Consider $\varphi = x \mathrel{-\!\!*} (y \Rightarrow (z \mathrel{-\!\!*} (w \Rightarrow k)))$. The graph $\mathscr{G}(\varphi)$ is



The blueprint of $\mathscr{G}(\varphi)$ can be seen as $\mathscr{G}(\varphi) - k$. In the notation of lemma 4.10, $M_1 = \{x\}$, $S_1 = \{y\}$, $M_2 = \{z\}$, and $S_2 = \{w\}$.